

The separate universe problem: 40 years on

B.J. Carr*

*School of Physics and Astronomy, Queen Mary University of London,
Mile End Road, London E1 4NS, UK and
Research Center for the Early Universe,
University of Tokyo, Tokyo 113-0033, Japan*

Tomohiro Harada†

Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan

(Dated: May 15, 2014)

Abstract

The claim that an overdense (positive curvature) region in the early universe cannot extend beyond some maximum scale and remain part of our universe, first made 40 years ago, has recently been questioned by Kopp *et al.* Their analysis is elucidating and demonstrates that one cannot constrain the form of primordial density perturbations using this argument. However, the notion of a separate-universe scale still applies and it places an important upper limit on the mass of primordial black holes forming at any epoch. We calculate this scale for equations of state of the form $p = k\rho c^2$ with $-1 < k < \infty$, refining earlier calculations on account of the Kopp *et al.* criticisms. For $-1/3 < k < \infty$, the scale is always of order the cosmological particle horizon size, with a numerical factor depending on k . This confirms the earlier claim that a primordial black hole cannot be much larger than the particle horizon at formation. For $-1 < k < -1/3$, as expected for some periods in the history of the universe, the situation changes radically, in that a sufficiently large positive-curvature region produces a baby universe rather than a black hole. There is still a separate-universe scale but the interpretation of these solutions requires care.

PACS numbers: 04.70.Bw, 97.60.Lf, 95.35.+d

*B.J.Carr@qmul.ac.uk

†harada-at-rikkyo.ac.jp

I. INTRODUCTION

It is often claimed that an overdense region in an expanding Friedmann-Robertson-Walker (FRW) background cannot extend too far, else it would be a separate closed universe rather than a region in our universe. This was first argued by Carr and Hawking [1] forty years ago and is particularly relevant to the formation of a primordial black hole (PBH) from an initial inhomogeneity because the region has to be quite close to the separate-universe limit in order to collapse against the pressure (at least in a radiation-dominated era).

A simple way to understand this is to regard the overdense region as part of a closed ($K = +1$) FRW model in a flat ($K = 0$) FRW background [2]. If the size of the region is big enough to include the entire closed FRW model, then it clearly cannot be part of *our* universe. If one considers the region at maximum expansion, when it has some density ρ_m , then the curvature of the spacelike hypersurface must be of order $G\rho_m/c^2$ and so the limiting scale is of order $c(G\rho_m)^{-1/2}$. The density ρ_m exceeds the density of the flat FRW background at that time by a factor Δ_m which depends on the equation of state. If this has the form $p = k\rho c^2$ for some constant k , then Δ_m is a simple function of k and the separate-universe scale is just the particle horizon size times a numerical factor which depends on this. The precise expression for $k > 0$ was first calculated by Harada and Carr [3]. The minimum size for a PBH is the Jeans length, which is a factor \sqrt{k} smaller than the particle horizon size, so PBHs can only form over a narrow range of scales unless k is very small (as in a dust era).

The above argument is simplistic because a real overdense region would not have a constant density profile (as required for a closed FRW model) and would need to be surrounded by an underdense region (as required by the flat FRW background assumption). It is also unrealistic to model a region which evolves into a PBH as a uniform overdensity. However, at least in this context more realistic calculations – which assume some form of density profile in the overdense region – have been performed [4, 5] and these roughly support the simplistic analysis. Nobody has yet done the equivalent calculation for the separate universe problem, even though the two problems are closely related.

Another point is that, although one can calculate the separate-universe condition by considering an overdense region at maximum expansion, this is a rather strange way of doing the calculation. If the overdense region is a separate universe at maximum expansion, it always was and always will be a separate universe. It does not *evolve* into a separate universe

because the topology cannot change in classical relativity. So there should be a more direct way of doing the calculation which is time-independent and just depends upon the total energy perturbation. However, since one is dealing with fluctuations on scales initially much larger than the particle horizon, one has to be careful which measure of the perturbation is used since this is gauge-dependent [6]. The analysis in Ref. [3] uses the *synchronous* gauge, which is convenient if one considers overdense regions at maximum expansion.

Recently, the separate-universe problem has been revisited and elucidated by Kopp *et al.* [7] (KHW). In particular, they describe the situation in terms of the *curvature* perturbation and are not restricted to analysing the problem at the time of the overdense region's maximum expansion. They provide some important new conceptual insights into the problem. In particular, if one considers overdense regions of increasing size at a given epoch, they emphasize that the Misner-Sharp mass reaches a maximum and then decreases to zero at the separate-universe scale. The density fluctuation also goes to zero there, although the curvature perturbation diverges. In this sense, they claim that the separate-universe condition is naturally avoided, with the strongest constraint on the density fluctuations coming from a scale which is smaller by a factor of two. Fluctuations are described as 'type I' and 'type II' according to whether they are smaller or larger than this scale.

KHW criticize the Carr-Hawking (CH) analysis, and also implicitly the Harada-Carr (HC) analysis, on this basis. The original CH analysis was certainly simplistic, so it is surprising that it has gone unchallenged for 40 years. In particular, neither CH nor HC differentiated between type I and type II fluctuations and they did not appreciate that the mass of the overdense region tends to zero and the curvature perturbation diverges as one approaches the separate-universe scale. This means that one cannot constrain primordial density perturbations, as originally claimed by CH. This is an important insight which changes one's interpretation of the problem. The use of embedding and conformal diagrams by KHW is also illuminating.

Nevertheless, we have a somewhat different perspective of the problem. We agree that there are some inadequacies in the CH and HC analyses and part of the purpose of this paper is to remedy these. However, we argue that the concept of separate-universe scale still applies and that the HC analysis provides the correct expression for this. Indeed, this expression has an important physical implication since it relates to the maximum mass of a PBH forming at any epoch in the history of the universe.

It must be stressed that our previous analysis only applied for $k > 0$ (although the expression we gave is valid for $k > -1/3$), while that of KHW is even more restricted, just focussing on the dust ($k = 0$) and radiation ($k = 1/3$) cases. On the other hand, the observed acceleration of the universe [8] can only be explained if the universe is dominated by some form of ‘dark energy’ – either a quintessence field [9] or a perfect fluid with $p = k\rho c^2$ and $k < -1/3$ [10]. This would also have applied during any inflationary period in the early universe. The analysis in Ref. [3] does not apply in these situations. Indeed, the whole concept of a small initial overdensity growing until it reaches some maximum expansion fails because density fluctuations *decay* with time for $k < -1/3$. So if overdense regions or black holes really can exist in such a universe, one essentially has to put them in at the outset (or at least the non-linear inhomogeneities which lead to them).

This problem is topical because recently a one-parameter family of spherically symmetric self-similar solutions has been found which contains black holes in an asymptotically FRW background with $k < -1/3$ [11, 12]. This means that the black hole grows at the same rate as the cosmological horizon (interpreted as the Hubble horizon since there is no particle horizon in this case). What is surprising about these solutions is that the black hole cannot be too *large* compared to the cosmological horizon: when the ratio of the sizes is too big (above 0.7 in the $k = -2/3$ case), one gets a cosmological *wormhole* (which connects two exact FRW models) instead of a black hole [13]. The transition occurs when the black hole and cosmological apparent horizons merge. By contrast, in the positive pressure case ($k > 0$), the black hole cannot be too *small* because the solution must be supersonic everywhere. The minimum size roughly corresponds to the Jeans condition.

There are various other differences between these solutions. One is that, while the black hole solutions are only asymptotically quasi-FRW in the $k > -1/3$ case, because there is a solid angle deficit at infinity [10], they are exactly asymptotically FRW in the $k < -1/3$ case. Another is that in $k > -1/3$ case the self-similar black hole solutions tend to a ‘universal black hole’ (in the sense that there is no cosmological particle horizon or black hole event horizon) as the parameter which describes them (a measure of the overdensity at large distances) gets sufficiently large. However, there is no such limiting solution in the wormhole case and this raises the question of whether wormholes replace separate universes for $k < -1/3$ or merely represent an *intermediate* situation. We also need to know whether this feature is specific to the self-similar situation or applies more generally. In any case,

the nature of the separate universe condition in the $k < -1/3$ situation clearly needs to be clarified. In an accompanying paper [14], we discuss the link between black holes and wormholes in a cosmological background in more detail.

The plan of this paper is as follows. Sec. II presents some general considerations, summarizing our previous approach to the separate-universe problem and emphasizing the ways in which it needs to be improved. Sec. III covers the $k > -1/3$ case and generalizes previous treatments of the problem. Sec. IV covers the more challenging $k \leq -1/3$ case, which has not been treated before. Sec. VI highlights some further aspects of the problem which remain to be explored. We make a more detailed comparison with the KHW analysis in Appendix A and include some mathematical details in Appendix B.

II. REASSESSING THE ARGUMENT FOR A SEPARATE-UNIVERSE SCALE

In this section, we first recall the usual argument for a separate-universe scale. We then discuss various inadequacies in this argument, many of them pointed out by KHW. Finally, we present a more rigorous analysis, in which the concept of a separate-universe scale is still shown to apply. We have no technical disagreement with the KHW analysis but we reformulate the problem in a way which relates more closely to the analysis of CH and HC.

A. General considerations

We take the cosmological background to be a flat ($K = 0$) FRW model with equation of state $p = k\rho c^2$, in which the line element is given by

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta)]. \quad (2.1)$$

Here r is a dimensionless comoving radial coordinate and the scale length $a(t)$ obeys the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3}, \quad (2.2)$$

with a dot denoting d/dt . This implies the background scale length and density evolve as

$$a_b \propto t^{\frac{2}{3(1+k)}}, \quad \rho_b = \frac{1}{6\pi G(1+k)^2 t^2}, \quad (2.3)$$

where t is the time since the big bang. The size of the background particle horizon (i.e. the proper radial distance travelled by light since the big bang) is

$$R_{PH}(t) = a(t) \int_0^t \frac{cdt}{a(t)} = \frac{3(1+k)}{1+3k} ct \quad (k > -1/3). \quad (2.4)$$

However, this expression diverges as $k \rightarrow -1/3$ and is inapplicable for $k < -1/3$, so it is generally more useful to refer to the Hubble horizon:

$$R_H(t) = \frac{c}{H} = \frac{3(1+k)}{2} ct \quad (k > -1) \quad (2.5)$$

where $H = \dot{a}/a$. Both horizons scale as t but they have a different k -dependence and are equal only for $k = 1/3$. The Hubble horizon is smaller than the particle horizon for $k < 1/3$ and only diverges as $k \rightarrow -1$. Since the mass within radius R is $M = 4\pi\rho R^3/3$, Eq. (2.2) implies

$$\frac{2GM}{Rc^2} = \frac{8\pi G\rho R^2}{3c^2} = \left(\frac{R}{R_H}\right)^2, \quad (2.6)$$

so the Hubble horizon is also equivalent to the cosmological *apparent* horizon. However, this only applies in a flat FRW model, when there is no ambiguity in the interpretation of the quantities R and M .

In order to describe the growth of an overdense region, one treats it as a homogeneous sphere which is part of a closed ($K = +1$) FRW model, an overdense region necessarily having positive curvature. The Friedmann equation then becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{c^2}{a^2}. \quad (2.7)$$

From Eqs. (2.2) and (2.7), one can deduce the first order equation for the evolution of a small density perturbation relative to the flat FRW background. This is defined as

$$\delta \equiv \frac{\rho - \rho_b}{\rho_b} \quad (2.8)$$

with respect to the synchronous gauge and evolves according to

$$\ddot{\delta} + \frac{4\dot{\delta}}{3(1+k)t} - \frac{2(1+3k)\delta}{3(1+k)t^2} = 0. \quad (2.9)$$

This gives a general solution

$$\delta = At^{\frac{2(1+3k)}{3(1+k)}} + Bt^{-1} \quad (2.10)$$

where A and B are constants. We took $B = 0$ in our previous analysis because only the A term represents a curvature perturbation; the decaying B term merely corresponds to a shift in the big bang time t_B , such a shift implying

$$\delta = \left(\frac{t}{t - t_B} \right)^2 - 1 \approx \frac{2t_B}{t} \quad (t \gg t_B). \quad (2.11)$$

However, the separate-universe condition can be modified by this effect if there is a curvature perturbation as well, so we extend the present analysis to include the B term. We will want to apply Eq. (2.10) for all values of k in the range -1 to $+\infty$, since these are physically well motivated. Both terms decay with time for $k < -1/3$ but the first term decays more slowly for $k > -5/9$.

B. Original heuristic analysis

We first focus on the $k > -1/3$ case and present an approximate argument of the kind used in Ref. [2]. At some initial time t_o , when the horizon mass is M_{Ho} , let the density fluctuation in the positive curvature region extend to a scale R_o and have amplitude δ_o . If we assume that the region stops expanding when $\delta \sim 1$, then Eq. (2.10) implies that the time at which this happens and its size and the horizon size then are given by

$$t_m \sim t_o \delta_o^{-\frac{3(1+k)}{2(1+3k)}}, \quad R_m \sim R_o \delta_o^{-\frac{1}{1+3k}}, \quad R_H(t_m) \sim R_{Ho} \delta_o^{-\frac{3(1+k)}{2(1+3k)}}. \quad (2.12)$$

(The subscript m indicates that these conditions pertain when the radius of the region attains its ‘maximum’.) The precise type of horizon (particle or Hubble) is not crucial for the rough analysis presented here but it was taken to be the particle horizon in the original papers. Note that the terms involving δ_o in Eq. (2.12) are large since δ_o is small and all the exponents are negative for $k > -1/3$.

At maximum expansion, the spatial hypersurface has a radius of curvature of order $c(G\rho_m)^{-1/2}$ and the Friedmann equation implies that this is around the horizon size. Therefore the region would close upon itself and form a separate universe if it were larger than this [1]. This situation is avoided providing

$$R_o/R_{Ho} < \delta_o^{-1/2}, \quad (2.13)$$

giving a limit on the initial density perturbation as a function of scale and mass:

$$\delta_o < (R/R_{Ho})^{-2} \sim (M/M_{Ho})^{-2/3}. \quad (2.14)$$

This is the limit first obtained in Ref. [2]. On the other hand, for collapse to a black hole, R_m must exceed the Jeans length, which is usually taken to be $R_J \approx \sqrt{k} R_H$, so PBH formation requires

$$(M/M_{Ho})^{-2/3} > \delta_0 > k(M/M_{Ho})^{-2/3} \quad (2.15)$$

and the density perturbation needs to be finely tuned [2].

It is also useful to express this result in terms of the conditions when the positive-curvature region enters the horizon. From Eq. (2.4), this happens at a time and density contrast

$$t_H \sim t_0(R_0/R_{H0})^{\frac{3(1+k)}{1+3k}}, \quad \delta_H \sim \delta_0(R_0/R_{H0})^2, \quad (2.16)$$

providing δ_H is small; this will turn out to be the condition for t_H to precede the maximum expansion epoch. We can now re-express Eq. (2.12) as

$$t_m \sim t_H \delta_H^{-\frac{3(1+k)}{2(1+3k)}}, \quad R_m \sim R(t_H) \delta_H^{-\frac{1}{1+3k}}, \quad R_H(t_m) \sim R(t_H) \delta_H^{-\frac{3(1+k)}{2(1+3k)}}, \quad (2.17)$$

where $R(t_H)$ is the size of the overdense region when it falls within the horizon; this is not to be confused with $R_H(t_m)$, which is the horizon size at t_m . To avoid a separate universe but collapse against the pressure, one therefore requires

$$1 > \delta_H > k, \quad (2.18)$$

although this means that the density perturbation at horizon crossing is not strictly in the linear regime. Subsequent numerical work in the radiation case ($k = 1/3$) suggests that this estimate for the lower limit is quite accurate, despite all the approximations involved. While earlier work gave a value of order 0.7 [15], more recent studies have reduced this to the range $0.3 - 0.5$ [16] and calculations of PBH formation in the context of critical phenomena give the range $0.43 - 0.47$ [17]. A recent refinement of the analytical constraint (2.18), which is in a better agreement with the numerical results, gives 0.41 in the radiation case [18].

C. Problems with heuristic analysis

In a more rigorous analysis, the significance of almost every quantity in the above analysis (R , δ_o , δ_m , M , R_H) needs to be considered more carefully. Besides affecting the final expression for the separate-universe scale, this has important conceptual implications. Some of the points discussed below were also raised by KHW.

Meaning of R . One needs to be mathematically precise in describing the geometry of an overdense region, especially when its size becomes comparable to the particle horizon. If we model it as part of a homogeneous closed FRW universe, then the line element (2.1) is replaced by

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (2.19)$$

where the scale length a now gives the curvature radius of the universe and we can choose $K = 1$. With this form of the metric, the area of the sphere of constant time is just $A = 4\pi a^2 r^2$, so we can define an ‘areal radius’ [19] by

$$R \equiv ar = (A/4\pi)^{1/2}. \quad (2.20)$$

The metric can also be expressed in the form

$$ds^2 = -c^2 dt^2 + a^2(t) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)], \quad (2.21)$$

where the dimensionless comoving radial coordinate χ is related to r by $r = \sin \chi$. For fixed t , this gives the “proper radius” (i.e. the proper length from the pole along a great circle of the 3-sphere), which we denote by L . Thus we have

$$R = a \sin \chi, \quad L = a\chi. \quad (2.22)$$

While R has a maximum value of a at $\chi = \pi/2$ and goes to 0 at $\chi = \pi$, L increases monotonically until it reaches the separate-universe scale $L_{max} = \pi a$. The relationship between these quantities is illustrated in Fig. 1. The apparent singularity in metric (2.19) at $r = 1/\sqrt{K}$, corresponding to $\chi = \pi/2$, is just a coordinate effect. The areal and proper radii are the same in a flat geometry but different in a closed geometry.

Meaning of M . The expression for the mass is ambiguous in the cosmological context, especially in the positive curvature case. There are various possible definitions even in the spherically symmetric situation. It is most convenient to use the Misner-Sharp mass, which can be expressed in terms of the areal distance R as

$$M = \frac{4\pi\rho R^3}{3}. \quad (2.23)$$

This goes to zero in the limit $\chi_a \rightarrow \pi$, essentially because the (negative) gravitational binding energy cancels the rest mass. As pointed out by KHW, it is meaningless to apply Eq. (2.14) in this limit, since the right-hand side diverges. One should therefore regard the

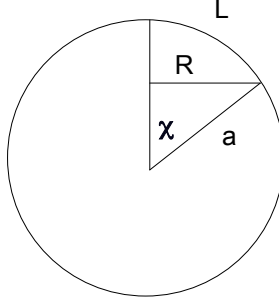


FIG. 1: This shows the relationship between the areal radius R , proper radius L and comoving coordinate χ in a closed FRW model at some time t .

separate-universe condition as a restriction on the size rather than the mass of the overdense region. Although the density associated with a black hole in asymptotically flat space scales as M^{-2} , which diverges as M approaches zero, M/R^3 is finite in the separate-universe limit, with a value corresponding to the curvature of the background Friedmann model.

Meaning of δ_o . There are different possible measures of the initial density perturbation on scales larger than the particle horizon. The quantity δ_o is defined by Eq. (2.8) as the fractional density excess relative to the flat FRW background, on the assumption that the density is homogeneous in the overdense region. However, the situation is more complicated than this, because the overdense region must be surrounded by a compensating void in order to ensure that the background is still flat. KHW therefore treat the overdense region as part of a positive-curvature FRW model with $0 \leq \chi \leq \chi_a$ and let this replace the part of the flat FRW background with $r \leq r_b$. There must be a vacuum region connecting these two parts but the pressure gradient caused by the matching is assumed to be small for super-horizon-scale perturbations. Outside the void there is no possibility of having a separate universe, so the issue only arises on scales within the void. In the separate-universe limit, the mass goes to zero anyway. KHW call perturbations ‘type I’ for $0 < \chi_a < \pi/2$ and ‘type II’ for $\pi/2 < \chi_a < \pi$. Any measure of the overall density perturbation (including δ_o) must depend upon χ_a in this simple model. KHW describe the overdensity in terms of a curvature perturbation and introduce two measures of this – a *central* volume fluctuation $\zeta(\chi_a)$ and an *averaged* volume fluctuation $\bar{\zeta}(\chi_a)$ – both being nearly time-independent on scales larger than the horizon. They also refer to a quantity $\delta(\chi_a)$, which represents the density perturbation when the overdense region enters the horizon. However, this is different from the (evolving)

quantity δ given by Eq. (2.10), so we write it as δ_{KHW} to avoid confusion. As the size of the region tends to the separate-universe scale, ζ and $\bar{\zeta}$ diverge but δ_o and δ_{KHW} go to zero. Although the two approaches should be equivalent, the KHW analysis has the advantage that it does not restrict attention to the time of maximum expansion.

Meaning of δ_m . Even if one analyses the separate-universe problem in terms of the size of an overdense region at maximum expansion, the previous heuristic approach is very imprecise. This is because the quantity δ is not exactly 1 at maximum expansion and the linear growth rate (2.10) only applies when δ is small. Both these inaccuracies can be accounted for by introducing the parameter

$$\Delta_m \equiv 1 + \delta_m \equiv \frac{\rho_m}{\rho_b(t_m)}, \quad (2.24)$$

which represents the ratio of the density in the overdense region at maximum expansion, ρ_m , to the density in the flat Friedmann background at the same time. Δ_m depends on the equation of state parameter k and the problem of how to calculate this dependence is treated later. This parameter enters the calculation because Eq. (2.7) implies that the maximum scale factor is

$$a_m = \sqrt{\frac{3c^2}{8\pi G\rho_m}} = \Delta_m^{-1/2} R_{Hb}(t_m), \quad (2.25)$$

where $R_{Hb}(t_m)$ is the background Hubble scale at t_m . From Eq. (2.5), the separate-universe scale at maximum expansion is

$$L_{max} = \pi a_m = \pi \Delta_m^{-1/2} R_{Hb}(t_m). \quad (2.26)$$

The proper radius associated with the separate-universe scale is always πa , but this is time-dependent, so it is useful to specify its maximum value.

Meaning of R_H . The expression (2.26) was obtained in Ref. [3] and also agrees with Eq. (16) of KHW. However, there is some arbitrariness about which horizon scale should be compared to L_{max} . Even if one uses the Hubble horizon rather than the particle horizon, there is still a difference between the *background* Hubble horizon used in Eq. (2.26) (equivalent to the background cosmological apparent horizon) and the *local* Hubble horizon (which is infinite at maximum expansion). The ratio of the separate-universe scale to the background Hubble scale always decreases for $k > -1/3$: it starts off large, goes as $a/t \propto t^{-(1+3k)/3(1+k)}$ at early times and falls to a value less than 2 at maximum expansion (see later). The ratio of the separate-universe scale to the local Hubble scale also decreases with time but goes

to zero at maximum expansion, so it is more useful to compare it to the local cosmological apparent horizon. The latter is always smaller than the background Hubble horizon [20] and is the relevant expression so long as this lies within the positive-curvature region. The ratio the separate-universe scale to this horizon is exactly π at maximum expansion [14].

D. More precise analysis

Let us now examine how the original heuristic expression for the separate-universe scale is modified if one includes all the corrections discussed above and assumes $k > -1/3$. To make contact with our earlier approach, we again assume that the positive-curvature region can be described by an initial overdensity δ_o , even though this is not the description used by KHW. Since $\rho \propto a^{-3(1+k)}$, Eq. (2.7) can be written as

$$\dot{a}^2 = Aa^{-(1+3k)} - B \quad (2.27)$$

where

$$A \equiv \frac{8\pi}{3}G\rho_o a_o^{3(1+k)} = [c^2 + \dot{a}_o^2]a_o^{1+3k} = \dot{a}_o^2\Omega_o a_o^{1+3k}, \quad B \equiv \dot{a}_o^2(\Omega_o - 1) = c^2. \quad (2.28)$$

Here the subscript o indicates some initial epoch at which the overdensity δ_o is small and

$$\Omega_o \equiv \frac{8\pi G\rho_o}{3H_o^2} = (1 + \delta_o) \left(\frac{H_{ob}}{H_o} \right)^2. \quad (2.29)$$

We distinguish between the initial values of the Hubble constant in the overdense region (H_o) and the background (H_{ob}). At a general epoch, we can express Eq. (2.27) in the form

$$\left(\frac{H}{H_b} \right)^2 = \Omega_M - \Omega_K, \quad (2.30)$$

where we have introduced the epoch-dependent dimensionless parameters

$$\Omega_M = \frac{8\pi G\rho}{3H_b^2}, \quad \Omega_K = \frac{c^2}{a^2 H_b^2}, \quad (2.31)$$

From Eq. (2.29), $\Omega_M(t_o)$ differs from Ω_o only by the factor $(H_{ob}/H_o)^2$.

The decomposition on the right-hand-side of Eq. (2.29) is gauge-dependent. For $k > -1/3$, we can always choose a (uniform Hubble) gauge in which $H_{ob} = H_o$ initially. In this case, the ratio of the local density to the background density, $\Delta(t)$, always exceeds unity and initially

$$\Omega_o = \Delta(t_o) = 1 + \delta_o. \quad (2.32)$$

However, another choice of gauge would be possible where this need not be the case. Indeed, for $k < -1/3$, we will see later that a positive-curvature region collapses from a dispersed state and then bounces into expansion, so it is necessarily *underdense* relative to the flat FRW background close to the big bang.

Since the expression for B in Eq. (2.28) implies

$$H_o = \frac{c}{a_o \sqrt{\Omega_o - 1}}, \quad (2.33)$$

the initial size of the region is given by

$$\frac{L_o}{R_{Ho}} = \frac{\chi_a}{\sqrt{\Omega_o - 1}}, \quad \frac{R_o}{R_{Ho}} = \frac{\sin \chi_a}{\sqrt{\Omega_o - 1}}. \quad (2.34)$$

Here R_{Ho} is the initial *local* Hubble scale, which differs from the initial *background* Hubble scale R_{Hob} by the factor H_{ob}/H_o . Putting $\dot{a} = 0$ in Eq. (2.27) gives the size of the region at maximum expansion:

$$L_m = a_m \chi_a = a_o \chi_a \left(\frac{\Omega_o}{\Omega_o - 1} \right)^{\frac{1}{1+3k}} = L_o \left(\frac{\Omega_o}{\Omega_o - 1} \right)^{\frac{1}{1+3k}}, \quad (2.35)$$

$$R_m = a_m \sin \chi_a = a_o \sin \chi_a \left(\frac{\Omega_o}{\Omega_o - 1} \right)^{\frac{1}{1+3k}} = R_o \left(\frac{\Omega_o}{\Omega_o - 1} \right)^{\frac{1}{1+3k}}. \quad (2.36)$$

Since the overdensity at maximum expansion can be expressed as

$$\Delta_m = \frac{\rho_m}{\rho_o} \frac{\rho_{bo}}{\rho_{bm}} \frac{\rho_o}{\rho_{bo}} = \frac{\rho_m}{\rho_o} \left(\frac{t_m}{t_o} \right)^2 \Omega_o \left(\frac{H_o}{H_{ob}} \right)^2, \quad (2.37)$$

we can write the time of maximum expansion as

$$t_m = t_o \Delta_m^{1/2} \Omega_o^{\frac{1}{1+3k}} (\Omega_o - 1)^{-\frac{3(1+k)}{2(1+3k)}} \left(\frac{H_{ob}}{H_o} \right). \quad (2.38)$$

Condition (2.13) is then replaced by

$$\frac{L_o}{R_{Ho}} < \frac{\pi}{\sqrt{\Omega_o - 1}}, \quad (2.39)$$

which corresponds to putting $\chi_a = \pi$ in Eq. (2.34). From Eq. (2.33) this is just equivalent to the condition $L_o < \pi a_o$, as expected. In the uniform-Hubble gauge, the right-hand-side of Eq. (2.39) can be written as $\pi \delta_o^{-1/2}$, so this might be regarded as a more precise version of the heuristic limit (2.13). From Eq. (2.34) there is also a purely geometrical upper limit,

$$\frac{R_o}{R_{Ho}} \leq \frac{1}{\sqrt{\Omega_o - 1}}, \quad (2.40)$$

with equality for $\chi_a = \pi/2$, corresponding to the condition $R_o < a_o$, but this is distinct from the separate-universe constraint since R_o goes to zero at $\chi_a = \pi$.

If we adopt Eq. (2.32), corresponding to $H_{ob} = H_o$, and use the relation $R_o = a_o \sin \chi_a$, we can interpret Eq. (2.33) as specifying δ_o as a function of χ_a :

$$\delta_o = (R_{Ho}/R_o)^2 \sin^2 \chi_a, \quad (2.41)$$

this being zero at $\chi = 0$ and $\chi = \pi$. In this case, we can regard the factor R_{Ho}/R_o as representing the dependence of δ_o on t_o for given χ_a . As illustrated in Fig. 2, this factor increases with t_o , starting small for $t_o \ll t_H$ and becoming 1 at horizon-crossing ($t_o = t_H$). This contrasts with the original CH analysis, where the parameter δ_o is assumed independent of χ , and with the KHW analysis, where the parameter ζ is constant in time until gradients build up after horizon-crossing.

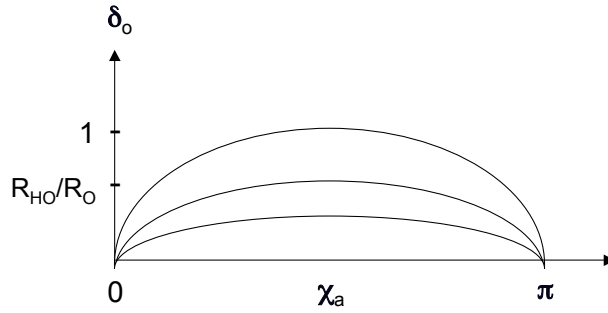


FIG. 2: This shows the dependence of the overdensity δ_o on χ_a . It is zero at $\chi = 0$ and $\chi = \pi$ and has a maximum value of R_{Ho}/R_o , which is 1 if the horizon epoch is taken as the initial time.

We can use Eq. (2.32) to express all our conditions in terms of δ_o . In particular, Eq. (2.12) is replaced by the *exact* relations

$$t_m = t_o \Delta_m^{1/2} \delta_o^{-\frac{3(1+k)}{2(1+3k)}} (1+\delta_o)^{\frac{1}{1+3k}}, \quad L_m = L_0 \left(\frac{\delta_o}{1+\delta_o} \right)^{-\frac{1}{1+3k}}, \quad R_{Hb}(t_m) = R_{Ho} \Delta_m^{1/2} \delta_o^{-\frac{3(1+k)}{2(1+3k)}} (1+\delta_o)^{\frac{1}{1+3k}}, \quad (2.42)$$

and condition (2.14) becomes

$$\delta_o < \left(\frac{\pi R_{Ho}}{L_o} \right)^2. \quad (2.43)$$

This differs from the original condition because it contains L_0/π rather than R_0 and because it is not expressed in terms of M . More importantly, Eq. (2.41) shows that this condition

reduces to

$$L_o/R_o < \pi/\sin \chi_a. \quad (2.44)$$

Since $R_o = a_o \sin \chi_a$ and $L_o = a_o \chi_a$ with $\chi_a < \pi$, this constraint is *necessarily* satisfied. Thus one still has a maximal proper length-scale for a fluctuation in our universe but there is no constraint on the fluctuation amplitude itself other than the purely geometrical condition $R_o < a_o$. In this sense, we agree with the conclusion of KHW that condition (2.14), even when expressed correctly, is not important.

Expressing the separate-universe condition in terms of the overdensity when the region falls within the horizon (i.e. the analogue of Eq. (2.17)) is problematic because we have seen that there is some ambiguity in what is meant by horizon-crossing. Since the areal radius rather than the proper radius is relevant for embedding a positive-curvature FRW model in a flat FRW background, we will regard horizon-crossing as occurring when the areal radius of the overdense region matches the background Hubble horizon. This has the important implication that a type II fluctuation enters the horizon very early, which means that a type II PBH still has a size comparable to the horizon at formation.

Applying Eq. (2.41) when $R_o = R_{Ho}$ implies

$$\delta_H = \sin^2 \chi_a. \quad (2.45)$$

However, R_{Ho} is the local Hubble scale and this is different from both the local cosmological apparent horizon scale R_{CAHo} and the background Hubble scale R_{Hob} . One can show that these quantities are related by [14]

$$R_{CAHo} = \Omega_{Mo}^{-1/2} R_{Hob} = \Omega_o^{-1/2} R_{Ho}, \quad (2.46)$$

so Eq. (2.41) gives

$$\delta_H = \frac{1}{2}(\sqrt{1 + 4 \sin^2 \chi_a} - 1) \approx \sin^2 \chi_a (1 - \sin^2 \chi_a). \quad (2.47)$$

This might be compared to the KHW expression in the radiation case,

$$\delta_{KHW} = \frac{1}{16} \sin^2 \chi_a (8 + \sin^2 \chi_a), \quad (2.48)$$

which differs by a factor $(8 + 9 \sin^2 \chi_a)/16$ for $\chi_a \ll 1$. (A simple analytic formula is only possible in this case.) The difference arises because they choose a gauge in which the time rather than the Hubble rate in the $K = 0$ and $K = +1$ FRW models are identified (i.e. they

use a synchronous rather than uniform Hubble gauge). Both expressions go to 0 as $\chi_a \rightarrow \pi$ and Eq. (2.17) is replaced with

$$t_m = t_H \Delta_m^{1/2} \delta_H^{-\frac{3(1+k)}{2(1+3k)}} (1+\delta_H)^{\frac{1}{1+3k}}, \quad R_m = R_H \delta_H^{-\frac{1}{1+3k}}, \quad R_{Hb}(t_m) = R(t_H) \Delta_m^{1/2} \delta_H^{-\frac{3(1+k)}{2(1+3k)}} (1+\delta_H)^{\frac{1}{1+3k}}. \quad (2.49)$$

The limit Eq. (2.43) becomes

$$\delta_H \leq \pi^2. \quad (2.50)$$

but again this condition is ensured by Eq. (2.45), so nothing new is learnt. The only disagreement with KHW concerns the divergence of the curvature perturbation at L_{max} . Since ζ cannot exceed infinity, KHW interpret this to mean the separate-universe condition cannot be achieved, whereas we interpret it to mean that ζ necessarily diverges!

Note that PBH formation is only possible if L_m exceeds the *local* Jeans length, which is a factor $\Delta_m^{1/2}$ smaller than the background Jeans length. However, the condition for PBH formation is still given by Eq. (2.15) because this cancels the factor of $\Delta_m^{1/2}$ for $R_{Hb}(t_m)$ in Eq. (2.49). It is interesting that the ratio of the Jeans scale to the separate-universe scale can exceed 1/2 for some values of k , implying that a PBH derives necessarily from a type II perturbation. The expression for the Jeans length adopted in Ref. [3] is

$$R_J = \frac{4\pi\sqrt{k}}{5+9k} cH^{-1} = 2\pi\sqrt{k} \left(\frac{1+3k}{5+9k} \right) R_{PH}, \quad (2.51)$$

where the last coefficient scales as \sqrt{k} for both $k \ll 1$ and $k \gg 1$. The exact expression is intriguingly close to half the separate-universe scale for all values of k between 0.1 and 1.

As pointed out by KHW, these considerations also have implications for the probability of PBH formation. One often assumes [2] that the horizon-epoch density fluctuations $\delta_H(M)$ have a Gaussian distribution as a function of M with some root-mean-square value $\epsilon(M)$. The probability of PBH formation for a given value of M then goes as

$$P(M) \sim \int_{\delta_{min}(M)}^{\delta_{max}(M)} \exp \left[- \left(\frac{\delta_H}{\epsilon(M)} \right)^2 \right] d\delta_H, \quad (2.52)$$

where $\delta_{max}(M)$ is associated with the separate-universe condition. With the CH argument, the precise form of the upper limit does not matter and it might as well be taken to be infinity because of the exponential term. However, if $\delta \rightarrow 0$ at the separate-universe scale, the distribution cannot be exactly Gaussian and it is misleading to regard the upper limit as a function of M at all. As stressed by KHW, only the curvature perturbation ζ can be

expected to be Gaussian and in this case the upper limit becomes infinite. Perhaps the most important aspect of the separate-universe calculation is that it implicitly specifies any upper limit on the size of a PBH forming at any epoch, this being half the separate-universe scale. Note that in the inflationary scenario there may also be an intrinsic non-Gaussianity in the distribution of δ_H [21] but we neglect this effect here.

E. Extending the analysis to the $k < -1/3$ case

In the $k \leq -1/3$ case, discussed in detail in Sec IV, the situation is fundamentally different in a number of respects. First, Eq. (2.10) implies that density inhomogeneities do not grow with time, so the concept of a slightly overdense region reaching some maximum expansion is inapplicable. Second, there is no particle horizon in this case, so one must use the Hubble horizon instead. Third, the significance of the factor Δ_m is different, since R may be momentarily static *before* t_0 and corresponds to a minimum rather than a maximum. The behaviour of the relativistic parameter $2GM/c^2R \propto a^{-(1+3k)}$ is similar in both the $k > -1/3$ and $k < -1/3$ cases: it first decreases from a large value, reaches a minimum which is necessarily less than 1 and then increases to a large value. However, the physical interpretation is different: for $k > -1/3$, any positive-curvature region first goes outside a local cosmological apparent horizon and later falls within a black hole apparent horizon (unless the pressure stops the collapse); for $k < -1/3$, it first goes outside a white hole apparent horizon and later falls within a local cosmological apparent horizon. We discuss the significance of this further in Ref. [14].

III. SEPARATE UNIVERSE CONDITION FOR $k > -1/3$

If we assume the overdense region is part of a closed FRW model with metric (2.21), then the largest proper radius of a 3-sphere at maximum expansion is π times the curvature radius a_m given by Eq. (2.25). Therefore it only remains to determine the overdensity at maximum expansion Δ_m . In the dust case, this is well known to be $(3\pi/4)^2$ and this result was generalized to the $1 > k > 0$ case in Ref. [3]. We reproduce our earlier argument here but extend it to include the $-1/3 < k < 0$ case and also allow for a variation in the big bang time. Some technical mathematical aspects of the calculation can be found in Appendix B.

For general k , the evolution of a is given by Eq. (2.27). By defining a new scale factor b and a new time coordinate τ such that

$$b = a^{1+3k}, \quad d\tau = (1+3k)b^{\frac{3k}{1+3k}} dt, \quad (3.1)$$

we can transform Eq. (2.27) into the dust form:

$$\left(\frac{db}{d\tau}\right)^2 = \frac{A}{b} - c^2. \quad (3.2)$$

Note that b and τ have dimensions $[L^{1+3k}]$ and $[TL^{3k}]$, respectively. This has the parametric solution

$$b = \frac{1}{2}b_m(1 - \cos \eta), \quad \tau = \frac{1}{\pi}\tau_m(\eta - \sin \eta), \quad \tau_m = \frac{b_m\pi}{2c}, \quad (3.3)$$

where $\eta = \pi$ corresponds to the epoch of maximum expansion (indicated by subscript m).

Equations (3.1) and (3.2) then imply

$$\begin{aligned} t &= \frac{1}{1+3k} \int_0^\tau b^{-\frac{3k}{1+3k}} d\tau \\ &= \frac{1}{1+3k} \left(\frac{b_m}{2}\right)^{-\frac{3k}{1+3k}} \left(\frac{\tau_m}{\pi}\right) \int_0^\eta (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta. \end{aligned} \quad (3.4)$$

This assumes that the overdense region has the same ‘big bang’ time as the background; we will examine the consequences of dropping this assumption later. Equations (2.27), (2.28) and (3.1) give

$$\frac{a_m}{a_o} = \left(\frac{\Omega_o}{\Omega_o - 1}\right)^{\frac{1}{1+3k}}, \quad \frac{b_m}{b_o} = \left(\frac{\Omega_o}{\Omega_o - 1}\right), \quad (3.5)$$

so we have

$$\frac{\rho_m}{\rho_o} = \left(\frac{a_m}{a_o}\right)^{-3(1+k)} = \left(\frac{\Omega_o - 1}{\Omega_o}\right)^{\frac{3(1+k)}{1+3k}}. \quad (3.6)$$

Since Eq. (2.3) implies that the background density at t_m is given by

$$\frac{\rho_{bm}}{\rho_{bo}} = \left(\frac{t_o}{t_m}\right)^2, \quad (3.7)$$

where t_0 and t_m are given by Eq. (3.4) with $\eta = \pi$ and $\eta = \eta_0$, respectively, Eq. (2.37) can be written as

$$\Delta_m = \left(\frac{\Omega_o - 1}{\Omega_o}\right)^{\frac{3(1+k)}{1+3k}} \left[\frac{\int_0^\pi (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta}{\int_0^{\eta_0} (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta} \right]^2 \Omega_o \left(\frac{H_o}{H_{ob}}\right)^2. \quad (3.8)$$

From Eq. (B7) of Appendix B, the top integral in Eq. (3.8) can be expressed as

$$\int_0^\pi (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta = 2^{\frac{1}{1+3k}} \sqrt{\pi} \frac{\Gamma\left(\frac{3(1+k)}{2(1+3k)}\right)}{\Gamma\left(\frac{2+3k}{1+3k}\right)}, \quad (3.9)$$

where Γ denotes the gamma function. From Eqs (3.3), (3.5) and (B8), the integral limit on the bottom is

$$\eta_o = \cos^{-1} \left(\frac{2 - \Omega_o}{\Omega_o} \right) \approx 2 \sqrt{\frac{\Omega_o - 1}{\Omega_o}} \left[1 + \frac{1}{6} \left(\frac{\Omega_o - 1}{\Omega_o} \right) \right], \quad (3.10)$$

where we assume $\Omega_o - 1 \ll 1$ in the last approximation, so that η_o is also small. Using Eq. (B9), the lower integral in Eq. (3.8) can then be approximated by

$$\int_0^{\eta_o} (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta \approx 2^{\frac{2+3k}{1+3k}} \left[\frac{1+3k}{3(1+k)} \right] \left(\frac{\Omega_o - 1}{\Omega_o} \right)^{\frac{3(1+k)}{2(1+3k)}} \left[1 + \frac{3(1+k)}{2(5+9k)} \left(\frac{\Omega_o - 1}{\Omega_o} \right) \right]. \quad (3.11)$$

The first term involving Ω_o cancels the equivalent term in Eq. (3.8) but a dependence on Ω_o still appears as a small correction in the second term. This gives

$$\Delta_m = \frac{\pi \alpha}{4} \left[\frac{3(1+k)}{1+3k} \right]^2 \frac{\Gamma \left(\frac{3(1+k)}{2(1+3k)} \right)^2}{\Gamma \left(\frac{2+3k}{1+3k} \right)^2} = \frac{\pi \alpha \Gamma \left(\frac{5+9k}{2(1+3k)} \right)^2}{\Gamma \left(\frac{2+3k}{1+3k} \right)^2} \quad (3.12)$$

where

$$\alpha \approx \left[1 + \frac{2(1+3k)}{5+9k} \left(\frac{\Omega_o - 1}{\Omega_o} \right) \right] \Omega_o \left(\frac{H_o}{H_{ob}} \right)^2 \quad (3.13)$$

is close to 1 and gives only a weak dependence on Ω_o . As k decreases from ∞ to $-1/3$, Δ_m increases from $(\pi/2)^2 \approx 2.3$ to ∞ , passing through 4 at $k = 1/3$ and $(3\pi/4)^2 \approx 5.3$ at $k = 0$.

Equation (2.26) now gives the separate universe scale in units of the background Hubble scale c/H_{bm} at maximum expansion:

$$c^{-1} H_{bm} L_{\max} = \pi \Delta_m^{-1/2} = 2 \sqrt{\frac{\pi}{\alpha}} \frac{(1+3k)}{3(1+k)} \frac{\Gamma \left(\frac{2+3k}{1+3k} \right)}{\Gamma \left(\frac{3(1+k)}{2(1+3k)} \right)} = \sqrt{\frac{\pi}{\alpha}} \frac{\Gamma \left(\frac{2+3k}{1+3k} \right)}{\Gamma \left(\frac{5+9k}{2(1+3k)} \right)}, \quad (3.14)$$

the factor of $5+9k$ in the last term relating to the ratio of the exponents in Eq. (2.10). This expression is plotted in Fig. 3. In a dust universe ($k = 0$), this gives $4/3$. In a radiation universe ($k = 1/3$), it gives $\pi/2$. In a (possibly unphysical) universe with $k = \infty$, it gives 2. In the limiting case $k = -1/3$, it goes to zero but we discuss this more carefully later. Note that L_{\max} exceeds the cosmological particle horizon for $k \gtrsim 0.126$ and the cosmological apparent horizon (or Hubble horizon) for $k \gtrsim -0.2$. For comparison, the maximum areal radius of an overdense region, associated with the maximum size of a PBH, is smaller by a factor of π . Although we have presented our analysis in terms of the evolution of an overdense region, it should be stressed that a region which is a separate universe at maximum expansion is *always* a separate universe.

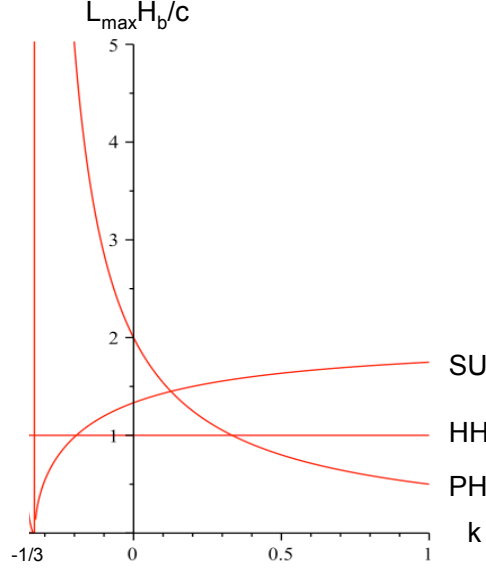


FIG. 3: This shows the separate-universe scale (SU), Hubble horizon scale (HH) and particle horizon scale (PH) in units of the background Hubble length as a function of the equation of state parameter k when the time of the big bang is unperturbed. The maximum size of a PBH is half the SU size.

In principle, the perturbed region could have a big bang time t_B which is different from the background universe. This corresponds to the decaying term Bt^{-1} in Eq. (2.10). One has various possible situations as summarized in Fig. 4. The previous analysis applies when $t_B = 0$ (case a). Providing $t_B < t_m$, so that the big bang in the background occurs before maximum expansion in the overdense region (case b), we can choose t_0 to be after t_B and then replace Eq. (3.8) with

$$\Delta_m = \left(\frac{\Omega_o - 1}{\Omega_o} \right)^{\frac{3(1+k)}{1+3k}} \left[\frac{\int_0^\pi \xi(1 - \cos \eta)^{\frac{1}{1+3k}} d\eta - t_B}{\int_0^{\eta_o} \xi(1 - \cos \eta)^{\frac{1}{1+3k}} d\eta - t_B} \right]^2 \Omega_o \left(\frac{H_o}{H_{ob}} \right)^2. \quad (3.15)$$

Here the timescale ξ is given by

$$\xi = \frac{1}{c(1+3k)} \left(\frac{b_m}{2} \right)^{\frac{1}{1+3k}} = \frac{2^{-\frac{1}{1+3k}}}{(1+3k)H_0} (\Omega_o - 1)^{-\frac{3(1+k)}{2(1+3k)}} \quad (3.16)$$

and we have assumed $\Omega_o \approx 1$. Equation (3.15) can also be expressed as

$$\Delta_m = \Delta_m(0) \left(1 - \frac{t_B}{t_m} \right)^2 \left(1 - \frac{t_B}{t_o} \right)^{-2}, \quad (3.17)$$

where $\Delta_m(0)$ is the value of Δ_m when $t_B = 0$ (the original expression) and the last term corresponds to $(H_{ob}/H_o)^2$ and so effectively cancels the term $(H_o/H_{ob})^2$ in Eq. (3.15). The

second factor is the important one and shows that the overdensity Δ_m decreases (i.e. $H_b L_{\max} = \pi \Delta_m^{-1/2}$ increases) as t_B increases. Furthermore, Δ_m goes to zero (i.e. $H_b L_{\max}$ diverges) in the limit $t_B = t_m$ because the background density is then infinite when the overdense region reaches maximum expansion. From Eq. (3.9),

$$t_m = 2^{\frac{1}{1+3k}} \xi \sqrt{\pi} \frac{\Gamma\left(\frac{3(1+k)}{2(1+3k)}\right)}{\Gamma\left(\frac{2+3k}{1+3k}\right)} \sim (\Omega_0 - 1)^{-\frac{3(1+k)}{2(1+3k)}} (H_b L_{\max})^{-1} (1+k)^{-1} \quad (3.18)$$

and this decreases as k increases since all three factors do. For any value of t_B , this means that there will be a value of k at which $H_b L_{\max}$ diverges; t_m falls below t_B for larger k , so that $H_b L_{\max}$ is no longer defined (case c). For $t_B < 0$, the last factor in Eq. (3.17) exceeds 1, so the separate-universe scale is reduced for all k . This explains the form of the curves in Fig. 5. The deviations from the earlier form essentially arise because the factor H_b is changed.

Although one cannot apply the above analysis for $t_B > t_m$, this does not mean that one cannot specify a separate-universe scale in this situation – just that one cannot specify it in terms of the background Hubble scale at maximum expansion. After such a positive-curvature regions has attained its maximum expansion, it will begin to recollapse. If it is not too extended, it will virialize at some radius (half its maximum radius in the dust case) due to the developments of pressure gradients and shocks. In this case, one can regard the region as forming a static spherically symmetric ‘retarded core’ with average density $\rho_c \approx 8\rho_m$. However, this cannot represent a consistent solution of the Einstein equations if the region falls within its Schwarzschild radius and this requires that it be smaller than

$$R_{BH} = \left(\frac{3c^2}{8\pi G \rho_c} \right)^{1/2}. \quad (3.19)$$

The separate-universe scale is somewhat larger than this,

$$L_{SU} = \pi R_{BH}, \quad (3.20)$$

so a black hole again represents an intermediate state before the separate-universe limit.

Finally, the overdense region may collapse *before* the big bang (case d), which might be regarded as a separate universe in time rather than space. However, if the collapsing region is just part of a closed universe, it may again virialise at some radius. In this case, it might conceivably persist through the background big bang. This possibility is indicated by the broken horizontal line in Fig. 4(d). However, it is unclear whether this situation is physically realistic, which is why there is a question mark next to this line.

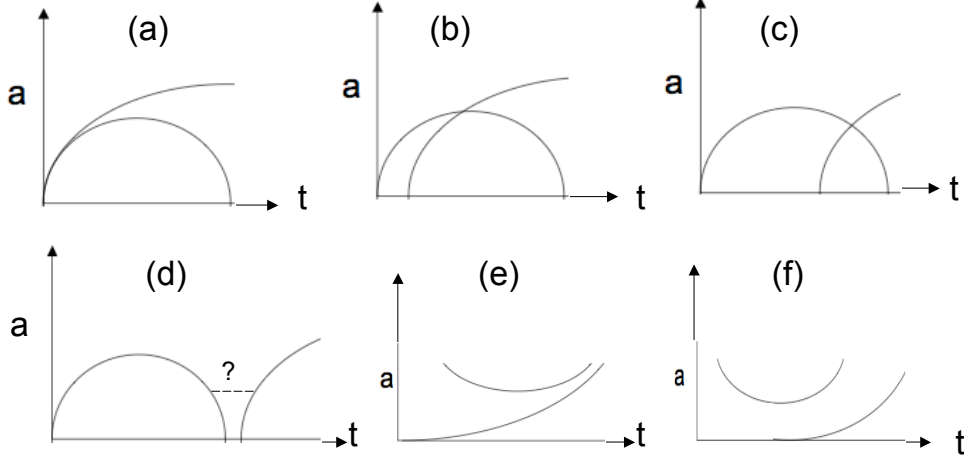


FIG. 4: This shows the evolution of the scale factor for the positive curvature region and background for $k > -1/3$ (cases a to d) and $k < -1/3$ (cases e and f) with various big bang times.

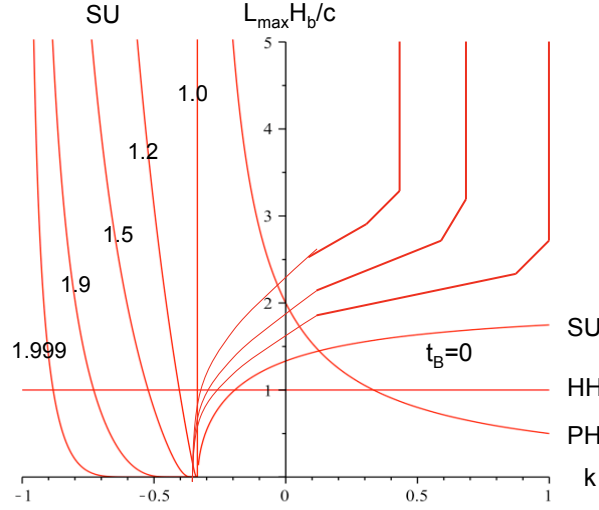


FIG. 5: As in Fig. 3 but allowing for a perturbed big bang time. The SU curves depend on the big bang delay in the background universe relative to the positive curvature region. For $k > -1/3$, the solid line corresponds to no delay ($t_B = 0$) and the other lines correspond to increasing delay. For $k < -1/3$, the curves are labelled by η_B/π where η is conformal time.

IV. SEPARATE UNIVERSE CONDITION FOR $k \leq -1/3$

A. $k \rightarrow -1/3^+$ limit

In the limit $k \rightarrow -1/3^+$, Eq. (3.11) becomes

$$\int_0^{\eta_o} (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta = \left(\frac{1+3k}{2} \right) \left[\frac{2(\Omega_o - 1)}{\Omega_o} \right]^{\frac{1}{1+3k}} \quad (4.1)$$

and this goes to 0 since the above analysis requires that Ω_o be close to 1, so that the term in square brackets is less than 1. In the same limit, Eq. (3.9) gives

$$\int_0^\pi (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta = 2^{\frac{1}{1+3k}} \sqrt{\pi}, \quad (4.2)$$

which necessarily diverges. Combining these asymptotic behaviours, Eq. (3.8) gives

$$\Delta_m \approx \frac{\pi}{(1+3k)^2}, \quad (4.3)$$

and Eq. (3.14) then implies

$$c^{-1} H_b L_{\max} = \sqrt{\pi} (1+3k), \quad (4.4)$$

corresponding to a separate-universe scale

$$L_{\max} = \frac{1}{2} \sqrt{\pi} (1+3k)^2 c t. \quad (4.5)$$

Both of these tend to 0 as $k \rightarrow -1/3^+$. This means that for the overdense region to stop expanding, the density contrast must get very high and hence the curvature radius of the overdense region is very small. The $k = -1/3$ case itself is clearly very special, because the first term in Eq. (2.10) is constant, so we consider that separately.

B. $k = -1/3$

For $k = -1/3$, the evolution of the positive-curvature Friedmann universe is particularly simple. From Eq. (2.7), for $8\pi G \rho_0 a_0^2 / (3c^2) = 1$, we find a *static* universe with

$$a = a_0, \quad \rho = \rho_0. \quad (4.6)$$

Since $L_{\max}(t) = \pi a(t)$ is constant, we have

$$c^{-1} H_b L_{\max} = \frac{\pi a_0}{t_b}, \quad \Delta = \frac{\rho}{\rho_b} = \frac{\rho_0}{\rho_{b0}} \left(\frac{t_b}{t_{b0}} \right)^2, \quad (4.7)$$

so $c^{-1}H_b L_{\max}$ decreases and the density contrast increases, despite there being no growing mode in the linear perturbation analysis. This indicates that such a perturbation must start nonlinear.

For $8\pi G\rho_0 a_0^2/(3c^2) > 1$, Eq. (2.7) implies

$$a = a_0 + \dot{a}_0(t - t_0), \quad \rho = \rho_0 \left(\frac{a}{a_0} \right)^{-2}, \quad (4.8)$$

with

$$\dot{a}_0 = \pm \sqrt{\frac{8\pi G\rho_0}{3}a_0^2 - c^2}. \quad (4.9)$$

In this case, the universe expands or collapses with a constant speed, depending on the initial conditions. The background Friedmann scale factor and density evolve as

$$a_b = a_{b0} \left(\frac{t_b}{t_{b0}} \right), \quad \rho_b = \rho_{b0} \left(\frac{a_b}{a_{b0}} \right)^{-2}, \quad (4.10)$$

where t_b is the cosmological time since the big bang. This implies

$$c^{-1}H_b L_{\max} = \frac{\pi(a_0 + \dot{a}_0(t - t_0))}{t_b}, \quad \Delta = \frac{\rho(t)}{\rho_b(t_b)} = \frac{\rho_0}{\rho_{b0}} \left(\frac{a(t)}{a_b(t_b)} \right)^{-2}. \quad (4.11)$$

We choose $t = t_b$, corresponding to the synchronous gauge, but the choice of t_0 is still arbitrary. (Note that the $8\pi G\rho_0 a_0^2/(3c^2) < 1$ case is prohibited.)

For the expanding case ($\dot{a}_0 > 0$), the density contrast Δ tends to a constant as $t = t_b \rightarrow \infty$. If we choose t_0 so that the big bang occurs simultaneously in the flat background and the positive-curvature region, which requires $t_0 = a_0/\dot{a}_0$, then Δ is always constant. This is consistent with the linear perturbation theory and implies

$$c^{-1}H_b L_{\max} = \pi\dot{a}_0, \quad (4.12)$$

where \dot{a}_0 is the (constant) positive root of Eq. (4.9). Since \dot{a}_0 is arbitrary, the separate universe length is also arbitrary. The assumption of a simultaneous big bang may be unnecessary if we consider initial quantum fluctuations which are nonlinear on very small scales. In this case, although $c^{-1}H_b L_{\max}$ and Δ are initially time-dependent, they tend to constants given by Eq. (4.12) as time proceeds. If the perturbed region is sufficiently large but smaller than the separate universe scale, this corresponds to a baby universe. Since it expands to large scales, it can eventually be described classically even though it is generated from quantum fluctuations.

For the collapsing case, it is not clear how to choose the origin of t with respect to the big bang. In general, ρ begins with some finite value and then monotonically increases to infinity. One might just exclude this possibility on the grounds that such a perturbation cannot evolve through the linear regime. On the other hand, such a perturbation might be generated on very small scales from quantum fluctuations, so we now pursue this possibility. If we assume $a_0 > 0$ and $\dot{a}_0 < 0$ at $t = t_0$ with $t_0 = 0$, then we find

$$c^{-1}H_b L_{\max} = \frac{\pi(a_0 + \dot{a}_0 t)}{t}, \quad \Delta = \frac{\rho(t)}{\rho_b(t)} = \frac{\rho_0}{\rho_{b0}} \left(\frac{a_0 + \dot{a}_0 t}{a_{b0} t} \right)^{-2} \quad (4.13)$$

and so $c^{-1}H_b L_{\max}$ starts infinite and then decreases monotonically to zero at $t = a_0/|\dot{a}_0|$, while the density contrast Δ starts zero and increases monotonically to infinity. This perturbation is nonlinear from the start and so can only be generated by quantum fluctuation on very small scales. If the perturbed region is sufficiently large but smaller than the separate universe scale, one has a black hole from the beginning and it is never classical.

C. $-1 < k < -1/3$

If we try to extend this analysis to the $k < -1/3$ regime, we immediately face the problem that small inhomogeneities decay rather than grow. Therefore the region must have been *initially* collapsing but then bounced into an expansion phase. The density perturbation can only be small well after that, so the ‘initial’ time t_0 must also be after the bounce. Nevertheless, we can still formally apply the previous analysis by extrapolating back to the time before t_0 at which the region was collapsing.

The analysis is different from that in the $k > -1/3$ situation for two reasons, both of which follow from Eq. (3.1). First, τ and t have different signs, which means that τ and hence η decrease as t increases. Second, $b \propto a^{1+3k}$, with the exponent of a being negative, so the *physical* scale factor a has a qualitatively different behaviour from the *mathematical* scale factor b . The positive-curvature region starts off collapsing from a dispersed state, bounces at some finite density and then ends up expanding into another dispersed state, so this gives the behaviour of $a(t)$. However, b starts at 0 when $\eta = 2\pi$, increases to a maximum at $\eta = \pi$ and then recollapses to 0 at $\eta = 0$. The crucial point is that the positive-curvature region does not grow from the background universe through a linear phase. Rather it is like having an extended (non-linear) object in a background universe from the start, analogous to the

‘retarded core’ considered in the $k > -1/3$ case.

Despite these differences, the separate universe analysis can still be carried out at the bounce epoch (i.e. at the moment of maximum density). Since the scale factor of the perturbed region has a minimum at the bounce, the separate universe scale also has a minimum then. In this situation, we can no longer take the lower integral limits in Eq. (3.8) to be 0 because the integrals diverge at this point (i.e. the scale factor a does not become infinite at a finite time). Instead, we must take the lower limit to be η_B with $0 < \eta_B < 2\pi$, where η_B is the value of η in the perturbed region corresponding to the big bang time of the background universe. We then replace Eq. (3.8) with

$$\Delta_m = \left(\frac{\Omega_o - 1}{\Omega_o} \right)^{\frac{3(1+k)}{1+3k}} \left[\frac{\int_{\eta_B}^{\pi} (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta}{\int_{\eta_B}^{\eta_o} (1 - \cos \eta)^{\frac{1}{1+3k}} d\eta} \right]^2 \Omega_o \left(\frac{H_o}{H_{ob}} \right)^2, \quad (4.14)$$

the first factor being larger than 1 and the second one smaller than 1. This expression only applies for $t_B < t_m$ (or $\eta_B > \pi$) since the overdensity is defined with respect to the background universe. Therefore we must assume $\eta_0 < \eta_B$ and $\pi < \eta_B < 2\pi$. As shown in Appendix B, the integrals have the form

$$\int_{\eta_B}^{\pi} (1 - \cos \eta)^{1/(1+3k)} d\eta \propto [A_k(\eta_B) - A_k(\pi)] \quad (4.15)$$

and

$$\int_{\eta_B}^{\eta_0} (1 - \cos \eta)^{1/(1+3k)} d\eta \propto [A_k(\eta_0) + A_k(\eta_B) - 2A_k(\pi)], \quad (4.16)$$

where the function A_k is given by Eq. (B3). The separate-universe scale at the bounce therefore becomes

$$c^{-1} H_b L_{\max} = \pi \Delta_m^{-1/2} = \pi \left(\frac{\Omega_o}{\Omega_o - 1} \right)^{\frac{3(1+k)}{2(1+3k)}} \left[\frac{A_k(\eta_0) + A_k(\eta_B) - 2A_k(\pi)}{A_k(\eta_B) - A_k(\pi)} \right]. \quad (4.17)$$

If we assume that t_0 is after t_m and that η_0 is sufficiently small for Eq. (B9) to apply, then

$$A_k(\eta_0) \approx 2^{3(1+k)/2(1+3k)} \left(\frac{\Omega_0 - 1}{\Omega_0} \right)^{3(1+k)/2(1+3k)}. \quad (4.18)$$

This term dominates $A_k(\eta_B)$ and $A_k(\pi)$ on the right-hand-side of Eq. (4.17) and so we obtain

$$c^{-1} H_b L_{\max} = \frac{2^{3(1+k)/2(1+3k)} \pi}{A_k(\eta_B) - A_k(\pi)}. \quad (4.19)$$

The dependence on Ω_0 disappears but the dependence on η_B remains. Using Eq. (B6) for $A_k(\pi)$, we finally obtain

$$c^{-1} H_b L_{\max} = 2\sqrt{\pi} \frac{(1+3k)}{3(1+k)} \frac{\Gamma\left(\frac{2+3k}{1+3k}\right)}{\Gamma\left(\frac{3(1+k)}{2(1+3k)}\right)} \left[\frac{A_k(\eta_B)}{A_k(\pi)} - 1 \right]^{-1}. \quad (4.20)$$

Apart from the factor in square brackets, this is the same as in the $k > -1/3$ case. The first factor implies $L_{\max} \rightarrow 0$ as $k \rightarrow -1/3^-$ and $L_{\max} \rightarrow \infty$ as $k \rightarrow -1$. Although the numerator diverges if $(2 + 3k)/(1 + 3k)$ is a negative integer, corresponding to $k = -2/3, -1/2, -4/9, \dots, -1/3$, Eq. (4.19) implies that the total expression is finite because $A_k(\eta_B) - A_k(\pi)$ is finite for $-1 < k < -1/3$. The form of $c^{-1}H_b L_{\max}$ for various values of η_B/π in the range 1 to 2 (i.e. for $t_B < t_m$) is shown in Fig. 5. Divergence occurs at $\eta_B = \pi$ (i.e. $t_m = t_B$) and this corresponds to the limit $k = -1/3$.

V. DISCUSSION

Prompted in part by the work of Kopp *et al.*, we have addressed various inadequacies in our previous treatment of the separate-universe problem. (1) We have compared our analysis with a more rigorous curvature perturbation approach. (2) We have allowed for the necessity of a compensated inhomogeneity in order to preserve the flat FRW background. (3) We have extended the analysis to include all equations of state with $-1 < k < \infty$ and find that the interpretation is very different for $k < -1/3$ and $k > -1/3$. (4) We have analysed the problem in terms of the Hubble horizon rather than the particle horizon and distinguished between the background and local horizons.

There are still several issues to be resolved. (1) We need to understand the effects of the density gradients in a more realistic situation. (2) We should relate our analysis to the self-similar black hole and wormhole solutions. (3) We should understand the significance of the self-similar universal black hole solutions (which have an apparent horizon but no event horizon). (4) We need to revisit the problem of black hole formation and consider the nature of the Jeans condition for $k < -1/3$. (5) We should determine whether a region bigger than the Jeans length is necessarily a type II fluctuation. (6) We should study the Penrose and embedding diagrams for these solutions carefully and relate this to the work of KHW. (7) In order to distinguish between separate universes, black holes, wormholes and baby universes, we must carefully analyse the behaviour of the trapped surfaces in these solutions and this problem is addressed in a separate paper [14].

The separate-universe condition can never be attained in any natural context (such as inflation) since the topology can never change in classical relativity but one might worry about it in the quantum gravitational context. Although the separate-universe condition

does not explicitly affect the PBH mass distribution, it does implicitly since the maximum mass of a PBH forming at any epoch is simply related to the separate-universe scale.

Acknowledgments

BC and TH are grateful to the Royal Society and JSPS for an exchange visitors grant. BC also thanks RESCEU, University for Tokyo, and TH thanks the Astronomy Unit, Queen Mary, University of London, for hospitality received during this work. TH was supported by the Grant-in-Aid for Young Scientists (B) (No. 21740190) and Challenging Exploratory Research (No. 23654082) for Scientific Research Fund of the Ministry of Education, Culture, Sports, Science and Technology, Japan and Rikkyo University Special Fund for Research.

Appendix A: Analysis of Kopp *et al.*

Here we compare our approach with the analysis of the separate-universe problem by KHW [7]. We first emphasize the points on which we agree. It is clearly sensible to describe the situation in terms of the curvature fluctuation (ζ) rather than the density fluctuation (δ), although we would claim that this is not a fundamental distinction. We also agree that it is important to distinguish between measuring the size of an overdense sphere in terms of the proper radial distance (L) and the areal radial distance (R). Indeed, we use their notation in the present paper to allow a direct comparison of our results and end up with a separate-universe condition corresponding almost exactly with theirs. One strength of the present work is that we allow the equation-of-state parameter k to vary over all possible values, including $k < -1/3$, whereas the KHW analysis is confined to the dust ($k = 0$) and radiation ($k = 1/3$) cases.

KHW describe the overdensity in terms of the curvature perturbation $\zeta(r, t)$, which is defined in terms of the perturbation to the spatial part of the metric (2.19) when expressed in the conformally flat form:

$${}^{(3)}ds^2 = b(t)^2 e^{2\zeta(r,t)} [dr^2 + r^2(d\theta^2 + \sin^2\theta)]. \quad (\text{A1})$$

They introduce three measures of this: a *central* volume fluctuation $\zeta(0, t)$, an *averaged* volume fluctuation $\bar{\zeta}(t)$, and the density fluctuation when the region falls within the Hubble

horizon δ_{KHW} . All three quantities can be expressed in terms of the extent of the overdense region χ_a . In the radiation case, they find

$$\zeta \approx -2 \ln \cos \frac{\chi_a}{2}, \quad \bar{\zeta} \approx \frac{1}{3} \ln \frac{2(\chi_a - \sin \chi_a \cos \chi_a)}{2 \sin^3 \chi_a}, \quad \delta_{KHW} = \frac{1}{16} \sin^2 \chi_a (8 + \sin^2 \chi_a), \quad (\text{A2})$$

and these three functions are plotted in their Figure (4). The first two expressions apply only so long as the region is much larger than the Hubble horizon; otherwise there are gradient effects and ζ and $\bar{\zeta}$ become time-dependent. Equation (A2) shows that ζ and $\bar{\zeta}$ diverge, while δ_{KHW} goes to zero, as the size of the region tends to the separate-universe scale ($\chi_a \rightarrow \pi$). KHW infer that the separate-universe condition can never be satisfied but there is still a separate-universe scale and the fact that ζ and $\bar{\zeta}$ diverge there merely indicates that something strange happens and not that this condition cannot be attained. Note that δ_{KHW} is different from what we term δ_H , which can be shown to be equivalent to 4ζ if $\chi_a \ll 1$.

KHW differentiate between Type I and Type II fluctuations but it is important to appreciate that one gets PBHs even in the type-II situation. It is interesting that the final PBH mass corresponding to χ and $\pi - \chi$ are the same and KHW usefully clarify what happens as $\chi \rightarrow \pi$, showing that $\delta \rightarrow 0$ in the limiting case. As χ increases, the Misner-Sharp mass reaches a maximum at $\chi = \pi/2$ and then shrinks to 0 as $\chi \rightarrow \pi$. This is why they claim that the separate-universe condition is necessarily avoided. They infer that one cannot place any constraint on the form of δ_0 and that the strongest constraint on the fluctuations comes from $\chi = \pi/2$ rather than $\chi = \pi$. We agree with this but in our view the fact that the mass tends to 0 as one approaches the separate-universe condition is just an equivalent way of stating the problem, albeit an illuminating one.

Appendix B: Hypergeometric function

The indefinite integrals of interest all have the form

$$\int^\eta (1 - \cos \eta)^{1/(1+3k)} d\eta = \pm \frac{\sqrt{2}(1+3k)}{3(1+k)} A_k(\eta) \quad (\text{B1})$$

where

$$A_k(\eta) \equiv (1 - \cos \eta)^{3(1+k)/2(1+3k)} F\left(\frac{1}{2}, \frac{3(1+k)}{2(1+3k)}, \frac{5+9k}{2(1+3k)}, \frac{1 - \cos \eta}{2}\right) \quad (\text{B2})$$

and $F(\alpha, \beta; \gamma; x)$ is the hypergeometric function. The positive sign applies for $0 < \eta < \pi$ and the negative sign for $\pi < \eta < 2\pi$. We can rewrite the last expression in the form

$$A_k(\eta) = (1 - \cos \eta)^\beta F\left(\frac{1}{2}, \beta; \beta + 1; \frac{1 - \cos \eta}{2}\right), \quad (\text{B3})$$

where $\beta \equiv 3(1+k)/2(1+3k)$. To evaluate the integral in the limits $\eta \ll 1$ (as required in the $k > -1/3$ case) and $\eta = \pi$ (as required in all cases), we recall the general result:

$$F(\alpha, \beta; \gamma; 0) = 1, \quad F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}. \quad (\text{B4})$$

In this case, $\alpha = 1/2$ and $\gamma = \beta + 1$, so we can rewrite the second relation as

$$F(1/2, \beta; \beta + 1; 1) = \frac{\Gamma(\beta + 1)\Gamma(1/2)}{\Gamma(\beta + 1/2)\Gamma(1)} = \frac{\sqrt{\pi}\Gamma(\beta + 1)}{\Gamma(\beta + 1/2)} \quad (\text{B5})$$

and express $A_k(\pi)$ in terms of gamma functions:

$$A_k(\pi) = \sqrt{\pi} 2^{3(1+k)/2(1+3k)} \frac{\Gamma\left(\frac{5+9k}{2(1+3k)}\right)}{\Gamma\left(\frac{2+3k}{1+3k}\right)}. \quad (\text{B6})$$

Although the two Γ terms diverge at particular values of k , their divergences cancel and the total expression is always finite. Equation (B1) then implies

$$\int^\pi (1 - \cos \eta)^{1/(1+3k)} d\eta = \pm 2^{\frac{1}{1+3k}} \sqrt{\pi} \frac{\Gamma\left(\frac{3(1+k)}{2(1+3k)}\right)}{\Gamma\left(\frac{2+3k}{1+3k}\right)}, \quad (\text{B7})$$

where we have simplified the expression using the relation $\Gamma(\beta + 1) = \beta \Gamma(\beta)$, which applies unless $\beta = 0$.

We also need an expression for $A_k(\eta_o)$ for $\eta_o \ll 1$, corresponding to some time when the density perturbation is small. In this case, the second order approximation

$$\cos^{-1} x \approx (6 - 2\sqrt{3+6x})^{1/2} \quad \text{with} \quad x = \frac{2 - \Omega_o}{\Omega_o} \quad (\text{B8})$$

leads to Eq. (3.10). We also have

$$\int^{\eta_o} (1 - \cos \eta)^{1/(1+3k)} d\eta \approx \int^{\eta_o} \left(\frac{\eta^2}{2}\right)^{1/(1+3k)} \left[1 - \frac{\eta^2}{12(1+3k)}\right] d\eta, \quad (\text{B9})$$

so integrating and using Eq. (3.10) gives Eq. (3.11). The first term also comes from Eq. (B2):

$$A_k(\eta_o) \approx \left(\frac{\eta_o^2}{2}\right)^\beta F\left(\frac{1}{2}, \beta; \beta + 1; 0\right) = \left(\frac{\eta_o^2}{2}\right)^{3(1+k)/2(1+3k)}, \quad (\text{B10})$$

which is small for $k > -1/3$ but large for $k < -1/3$. Special consideration needs to be given to the $k = -1/3$ and $k = -1$ cases.

-
- [1] B.J. Carr and S.W. Hawking, Mon. Not. R. Astron. Soc. **168**, 399 (1974).
 - [2] B.J.Carr, Astrophys. J. **201**, 1-19 (1975).
 - [3] T. Harada and B.J. Carr, Phys. Rev. D**71**, 104010 (2005).
 - [4] D. K. Nadezhin, I. D. Novikov and A. G. Polnarev, Sov. Astron. **22**, 129 (1978).
 - [5] I. Musco and A.G. Polnarev, Class. Quant. Grav. **24**, 1405 (2009).
 - [6] J. Bardeen, Phys. Rev. D**22**, 1882 (1980).
 - [7] M. Kopp, S. Hofmann and J. Weller, Phys. Rev. D**83** 124025 (2011).
 - [8] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); A.G. Riess *et al.*, Astron. J. **116**, 1009 (1998); Astron. J. **117**, 707 (1999).
 - [9] B. Ratra and J. Peebles, Phys. Rev. D**52**, 1837 (1995).
 - [10] B.J. Carr, T. Harada and H. Maeda, Class. Quant. Grav. **27**, 183101 (2010).
 - [11] T. Harada, H. Maeda and B.J. Carr, Phys. Rev. D**77**, 024022 (2008).
 - [12] H. Maeda, T. Harada and B.J. Carr, Phys. Rev. D**77**, 024023 (2008).
 - [13] H. Maeda, T. Harada and B.J. Carr, Phys. Rev. D**79**, 044034 (2009).
 - [14] B.J. Carr and T. Harada, preprint (2014).
 - [15] J. Niemeyer and K. Jedamzik, Phys. Rev. D**59**, 124013 (1999).
 - [16] A.M. Green, A.R. Liddle, K.A. Malik and M. Sasaki, Phys. Rev. D**70**, 041502 (R) (2004).
 - [17] I. Musco, J. Miller and L. Rezzolla, Class. Quant. Grav. **22**, 1405 (2005).
 - [18] T. Harada, C. M. Yoo and K. Kohri, Phys. Rev. D**88**, 084051 (2013).
 - [19] S.A. Hayward, Phys. Rev. D**53**, 1938 (1996).
 - [20] V. Faraoni, arXiv:1106.4427 (2011).
 - [21] J.S. Bullock and J.R. Primack, Phys. Rev. D**55** 7423 (1997); P. Ivanov, Phys. Rev. D**57** 7145 (1998); J.C. Hidalgo, arXiv:0708.3875 (2007); C.T. Byrnes, E.J. Copeland and A.M. Green, Phys. Rev. D**86** 043512 (2012).